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Two Critical Points in Bose Condensation of Trapped Cold Atoms

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- Bose-Einstein Condensation (BEC) and "Quasi-Condensate": Perfect Bose-gas
- Motivation: experimental "Quasi-Condensate" problem and two critical points
- Theoretical remarks about BEC and "Quasi-Condensate"
- Conclusion: BEC in 3D exponentially anisotropic "cigar" traps is the "Quasi-Condensate" with two critical points

Mathieu Beau and V.Z., Cond.Mat.Phys.**31**, 23003:1-10 (2010)

0. Perfect Bose-gas: Einstein (1925),Uhlenbeck (1927),London (1938),Casimir (1968), van den Berg-Lewis-Pulé (1978)...

• For $\Lambda = L_1 \times L_2 \times L_3 \in \mathbb{R}^3$ and $T_\Lambda^{(N=1)} = \left(-\hbar^2\Delta/(2m)\right)_D$ the spectrum:

$$\left\{\varepsilon_s = \frac{\hbar^2}{2m}\sum_{j=1}^{3}(\pi s_j/L_j)^2\right\}_{s_j\in\mathbb{N}}$$

• Eigenfunctions: $\{\phi_{s,\Lambda}(x) = \prod_{j=1}^3 \sqrt{2/L_j}\,\sin(\pi s_j x_j/L_j)\}_{s_j\in\mathbb{N}}$, $s := (s_1, s_2, s_3) \in \mathbb{N}^3$

• In (T, V, μ), $V = L_1L_2L_3$ the Gibbs mean occupation number of $\phi_{s,\Lambda}$ is $N_s(\beta,\mu) = (e^{\beta(\varepsilon_s-\mu)} - 1)^{-1}$, $\mu < \inf_s\ \varepsilon_s$.

• Particle density $\rho_\Lambda(\beta,\mu) = \sum_{s\in\mathbb{N}^3} N_s(\beta,\mu)/V =: N_\Lambda(\beta,\mu)/V$

• The **first** critical density: $\rho_c(\beta) := \sup_{\mu\leq 0}\lim_\Lambda \rho_\Lambda(\beta,\mu) = \zeta(3/2)/\lambda_\beta^3$, $\lambda_\beta := \hbar\sqrt{2\pi\beta/m}$, de Broglie thermal length.

- **Proposition 1.1** Generalized BEC $\neq$ Conventional BEC.

- [Uhlenbeck(1927)] *Dual set* Λ^* of momenta w.r.s. to the p.b.:

$$\Lambda^* := \{k_j := \frac{2\pi}{V^{\alpha_j}} n_j : n_j \in \mathbb{Z}\}_{j=1}^{d=3} \quad \text{and} \quad \varepsilon_k := \sum_{j=1}^{d} k_j^2/2$$

- **Cube**: $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, $V = L^3$. If $\mu < 0$ and $\Lambda \nearrow \mathbb{R}^3$:

$$\begin{aligned}
\rho &= \lim_{\Lambda} \rho_\Lambda(\beta,\mu) := \lim_{\Lambda} \frac{1}{V}\{\frac{1}{e^{-\beta\mu}-1} + \sum_{k\in\{\Lambda^*\setminus\{0\}\}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}\} \\
&= \lim_{L\to\infty} \frac{1}{L^3} \sum_{n_j\in\mathbb{Z}\setminus\{0\}} \left\{e^{\beta(\sum_{j=1}^{d}(2\pi n_j/V^{1/3})^2/2-\mu)} - 1\right\}^{-1} \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \left\{e^{\beta(k^2/2-\mu)} - 1\right\}^{-1} =: \mathfrak{I}(\beta,\mu).
\end{aligned}$$

Controversy:

- For $d > 2$ the *perfect* Bose-gas *critical density*

$$\rho_c(\beta) := \lim_{\mu \nearrow 0} \mathfrak{I}(\beta, \mu) \ ,$$

is finite !

- Then **if** $\rho > \rho_c(\beta) \Rightarrow$ **BEC** at $k = 0$ (ground-state) i.e. $n_1 = n_2 = n_3 = 0$:

$$\rho_0(\beta) := \rho - \rho_c(\beta) \ .$$

1.2 Saturation Mechanism (*conventional* **BEC** *condensation*):
[F.London(1938)] Let $\mu_\Lambda(\beta,\rho)$ be solution of the equation

$$\rho = \rho_\Lambda(\beta,\mu) \quad \Leftrightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\beta,\rho)) \qquad (scaling!).$$

- $\lim_\Lambda \ \mu_\Lambda(\beta, \rho < \rho_c(\beta)) = \mu_\Lambda(\beta,\rho) < 0$ or
- $\lim_\Lambda \ \mu_\Lambda(\beta, \rho \geq \rho_c(\beta)) = 0$, and

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_\Lambda \frac{1}{V} \ \left\{ e^{-\beta\, \mu_\Lambda(\beta,\rho\geq\rho_c(\beta))} - 1 \right\}^{-1} \Rightarrow$$

$$\mu_\Lambda(\beta, \rho \geq \rho_c(\beta)) = -\frac{1}{V} \frac{1}{\beta(\rho - \rho_c(\beta))} + o(1/V) \ .$$

- Since $\varepsilon_k = \sum_{j=1}^{d} (2\pi n_j / V^{1/3})^2/2$, the BEC is in **k=0**-mode:

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k\neq 0} - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} = 0 \ ,$$

- This is a well-known *conventional* (**type I**) BEC.

1.3 Saturation Mechanism (*generalised condensation*):

• **The Casimir Box (1968):** Let $\alpha_1 = 1/2$, i.e. $\alpha_2 + \alpha_3 = 1/2$. Since $\varepsilon_{k_1,0,0} = (2\pi n_1/V^{1/2})^2/2 \sim 1/V$, then again the solution of

$$\rho = \rho_\Lambda(\beta,\mu) \quad \Leftrightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\beta,\rho)).$$

has the asymptotics $\mu_\Lambda(\beta, \rho \geq \rho_c(\beta)) = -A/V + o(1/V),\ A \geq 0$, although the number of modes producing condensate is **infinite**:

$$\lim_\Lambda \left\{ \frac{1}{V}\frac{1}{e^{-\beta\mu_\Lambda(\beta,\rho)} - 1} + \frac{1}{V} \sum_{k\in\{\Lambda^*: n_1\neq 0, n_2=n_3=0\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1} \right\}$$
$$= \rho - \rho_c(\beta) > 0.$$

$$\lim_\Lambda \frac{1}{V}\left\{e^{\beta(\varepsilon_{k\neq 0} - \mu_\Lambda(\beta,\rho))} - 1\right\}^{-1} \neq 0, \text{ for } \varepsilon_{k\neq 0} = \varepsilon_{k_1,0,0} \sim \mu_\Lambda(\beta,\rho)\ ,$$

$$\lim_\Lambda \frac{1}{V}\left\{e^{\beta(\varepsilon_{k\neq 0} - \mu_\Lambda(\beta,\rho))} - 1\right\}^{-1} = 0, \varepsilon_{0,k_{2,3}\neq 0} \sim (2\pi n_j/V^{\alpha_j})^2/2 > \mu_\Lambda(\beta,\rho).$$

• Generalised BEC **type II** [van den Berg-Lewis-Pulé (1978)]:

$$\rho - \rho_c(\beta) = \lim_{L\to\infty} \frac{1}{V} \sum_{n_1 \in \mathbb{Z}} \left\{ e^{\beta((2\pi n_1/V^{1/2})^2/2 - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1}$$

$$= \sum_{n_1 \in \mathbb{Z}} \frac{1}{(2\pi n_1)^2/2 + A} \,.$$

Here $A \geq 0$ is a *unique root* of the above equation.

• **N.B.** For $\alpha_1 = 1/2$ the BEC is still mode by mode **microscopic**, but **infinitely** *fragmented =quasi-condensate.* Experiments with *rotating* condensate (2000) and *chaotic* phases (2008).

• **The van den Berg-Lewis-Pulé Box:** $\alpha_1 > 1/2$.

• **Proposition 1.2** No macroscopic occupation of **any(!)** level:

$$\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} = 0.$$

• Generalised BEC **type III** [van den Berg-Lewis-Pulé (1978)]: $\alpha_1 > 1/2$ i.e. $\alpha_2 + \alpha_3 < 1/2$.

• Since $\varepsilon_{k_1,0,0} = (2\pi n_1/V^{\alpha_1})^2/2 \sim 1/V^{2\alpha_1}$, $2\alpha_1 > 1$, then the solution $\mu_\Lambda(\beta, \rho)$ has **a new** asymptotics: $\mu_\Lambda(\beta, \rho \geq \rho_c(\beta)) = -B/V^\delta + o(1/V^\delta)$, with $B \geq 0$.

• To this end we first must consider the particle density due to summation in k_1-modes:

$$\frac{1}{V} \sum_{k \in \{\Lambda^*:(n_1,0,0)\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1} =$$

$$\frac{1}{V} \sum_{k \in \{\Lambda^*:(n_1,0,0)\}} \sum_{s=1}^{\infty} e^{-s\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} =$$

$$\frac{1}{V} \sum_{s=1}^{\infty} e^{s\beta\mu_\Lambda(\beta,\rho)} \sum_{n_1=0,\pm1,\pm2,\ldots} e^{-s\beta((2\pi)^2 n_1^2/2V^{2\alpha_1})}.$$

• **N.B.** We are not care very much about $\alpha_2 + \alpha_3 < 1/2$ and about summation over the modes k_2, k_3, since for $n_{2,3} \neq 0$

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1, n_2 \neq 0, n_3 \neq 0)\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1} = \rho_c(\beta), \;\; \rho > \rho_c(\beta).$$

the *Darboux-Riemann* integral-sum converges to $\rho_c(\beta)$.

• For k_1 summation we apply the *Jacobi identity*, with parameter $\lambda = s\beta 2\pi V^{-2\alpha_1}$:

$$\sum_{n_1 = 0, \pm 1, \pm 2, \ldots} e^{-\pi \lambda n_1^2} \equiv \frac{1}{\sqrt{\lambda}} \sum_{\xi = 0, \pm 1, \pm 2, \ldots} e^{-(\pi \xi^2 / \lambda)} \Rightarrow$$

$$\sum_{n_1 = 0, \pm 1, \pm 2, \ldots} e^{-s\beta((2\pi)^2 n_1^2 / 2V^{2\alpha_1})} = \frac{V^{\alpha_1}}{\sqrt{s\beta 2\pi}} \sum_{\xi = 0, \pm 1, \pm 2, \ldots} e^{-(\pi \xi^2 V^{2\alpha_1} / s\beta 2\pi)} \Rightarrow$$

therefore, only the $\xi = 0$ term survives in the limit $V \to \infty$!

- Thus for the generalized BEC density of the **type III** one obtains:

$$\rho - \rho_c(\beta) =$$
$$\lim_{\Lambda} \left\{ (2\pi\beta)^{-1/2} \left\{ \frac{V^{\alpha_1 - 1}}{V^{\delta/2}} \cdot V^{\delta} \right\} \frac{1}{V^{\delta}} \left\{ \sum_{s=1}^{\infty} e^{-\beta B\left(s/V^{\delta}\right)} \left(\frac{s}{V^{\delta}} \right)^{-1/2} \right\} \right\} .$$

- This limit is *nontrivial* only for $\delta = 2\,(1 - \alpha_1) < 1$:

$$0 < \rho - \rho_c(\beta) = (2\pi\beta)^{-1/2} \int_0^{\infty} d\xi \; e^{-\beta B \xi} \; \xi^{-1/2} \; .$$

- The parameter $B = B\,(\beta, \rho) > 0$ is the *unique* root of the equation:

$$\rho - \rho_c(\beta) = \frac{1}{\sqrt{2\beta^2 B(\beta, \rho)}} \; .$$

• **Generalised BEC of type III**: one-mode particle occupations:

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda (\beta, \rho > \rho_c(\beta))) = 0 \text{ for } \textbf{all } k \in \{\Lambda^*\} \ .$$

• For the *"renormalized"* k_1-modes occupation "density" one obtains:

$$\lim_{\Lambda} \frac{1}{V^{2(1-\alpha_1)}} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda (\beta, \rho > \rho_c(\beta))) = 2\beta \left(\rho - \rho_c(\beta)\right)^2 ,$$

where $k \in \{\Lambda^* : (n_1, 0, 0)\}$ and $2\,(1-\alpha_1) = \delta < 1$.

• **Definition 1.3** (generalised BEC)

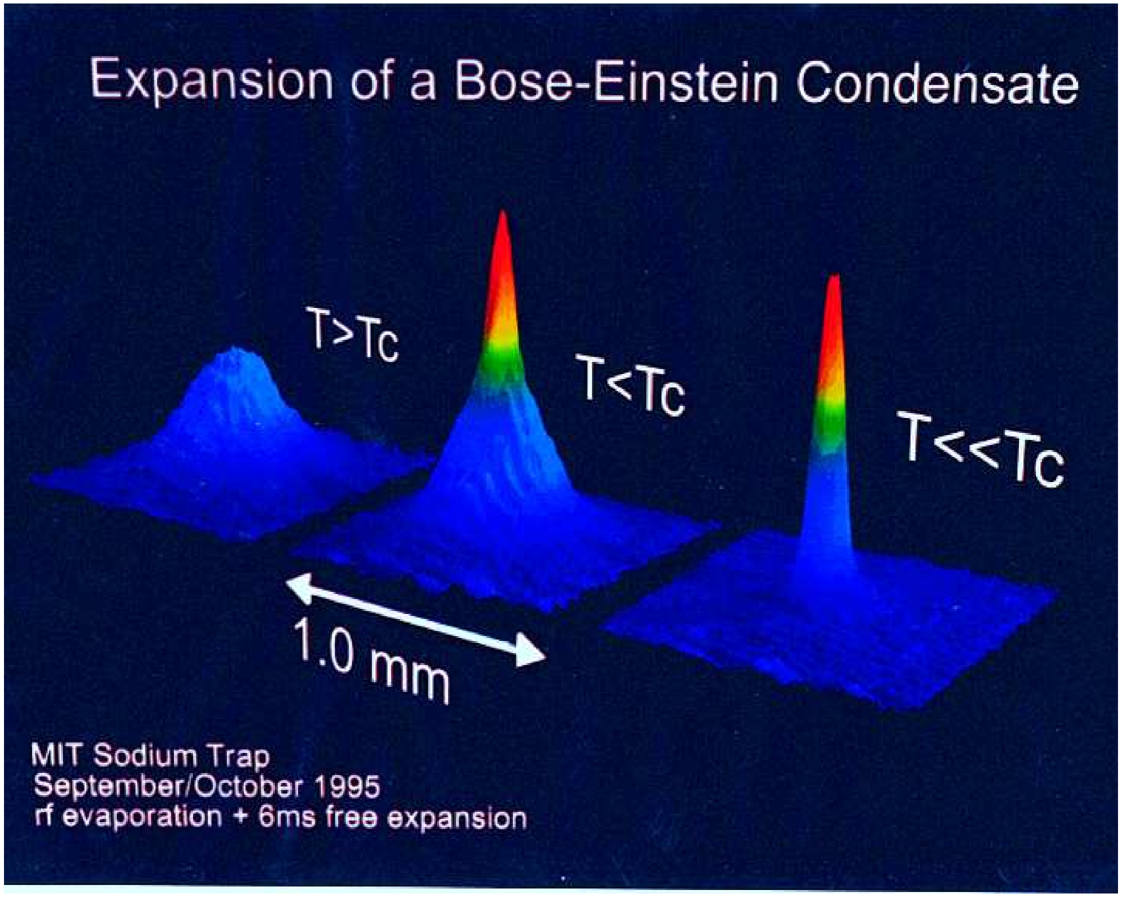
$$\rho - \rho_c(\beta) := \lim_{\eta \to +0} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| \leq \eta\}} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} \ .$$

• **Saturation ρ_m-PROBLEM**: [*van den Berg-Lewis-Pulé*] Is it possible that: $\rho_c \leq \rho_m \leq \infty$ such that **type III (or II)** $\rightarrow$ **type I**, for $\rho \geq \rho_m$? Yes! [Ch.2 BEC with the Second Critical Point].

1. Motivation: experimental "Quasi-Condensate" Problem

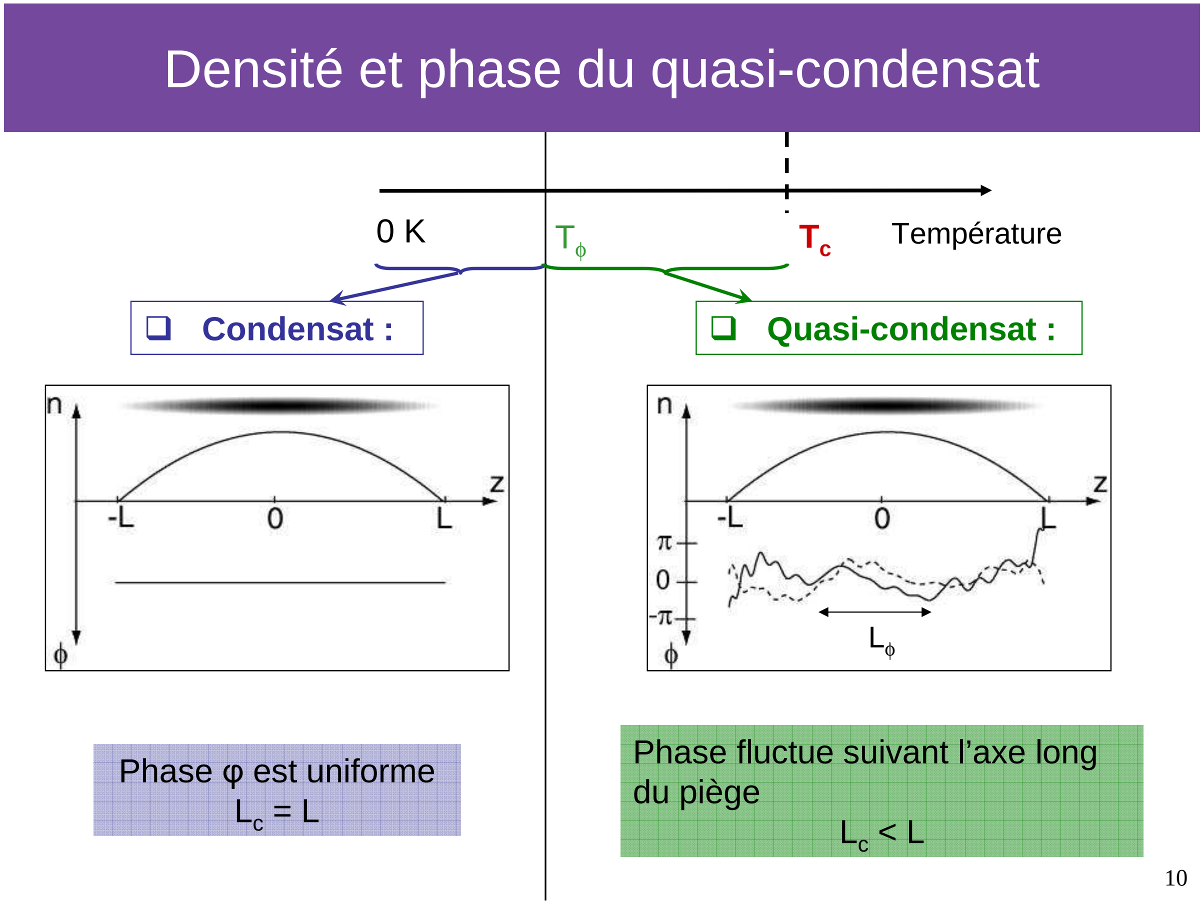
• Main Experimental Features I [PRL(2000-2003)]:

(a) In the *elongated* harmonic ("cigar") traps the **two** stages of the condensation were observed and associated with **two** temperatures T_c and $T_\phi (< T_c)$ in 2000-03.
(b) The properties of the condensate are changing drastically:
⇒ For $T_\phi < T < T_c$ the coherence along the "cigar" is weak and the condensate density and the phase are *strongly* fluctuating.
⇒ For $0 \leq T < T_\phi$ there is a strong coherence along the "cigar" and *weak* fluctuations of density and phase.
(c) In contrast to a uniform (one-phase) coherence of the *conventional* **condensate** there is a "chaotic" (multi-phase) internal structure: **"quasi-condensate"** for $T_\phi < T < T_c$.

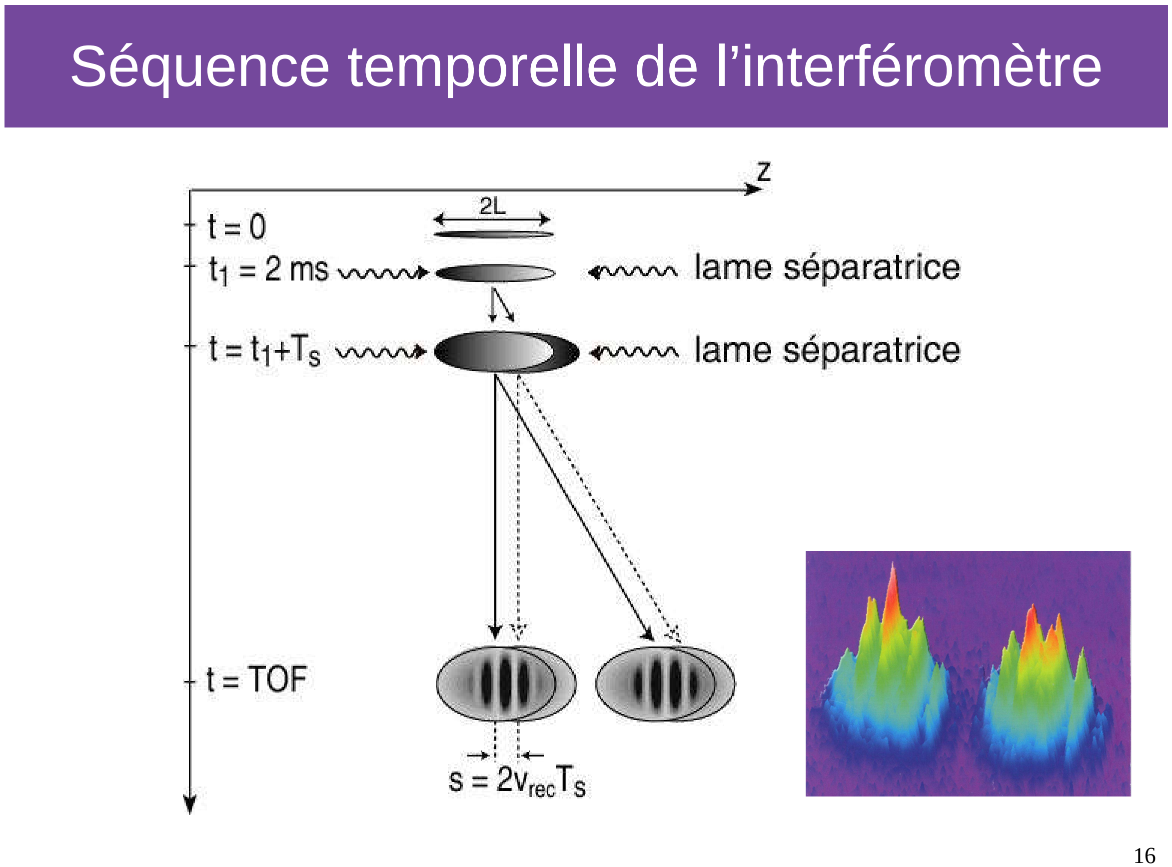

Expansion of a Bose-Einstein Condensate
T>Tc
T<Tc
T<<Tc
1.0 mm
MIT Sodium Trap
September/October 1995
rf evaporation + 6ms free expansion

BEC in trap

- **Main Experimental Features II [PRL(2000-2003)]:**



- **Main Experimental Features III (Internal Structure):**

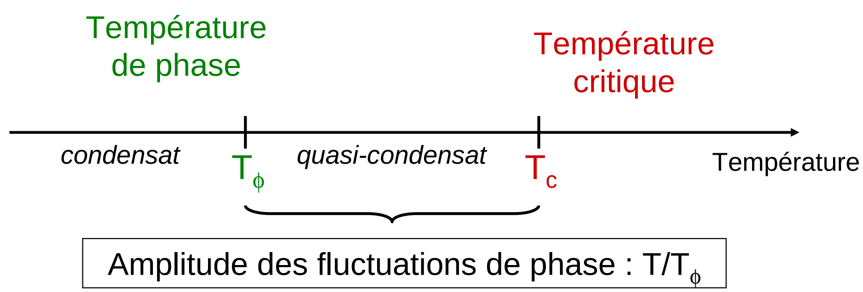


- **Main Experimental Conclusions:**

1. Existence of **TWO** critical points.
2. Existence of a **NEW** type of condensate: "Quasi-Condensate".

Résumé

- Deux températures pour la caractérisation de la condensation :



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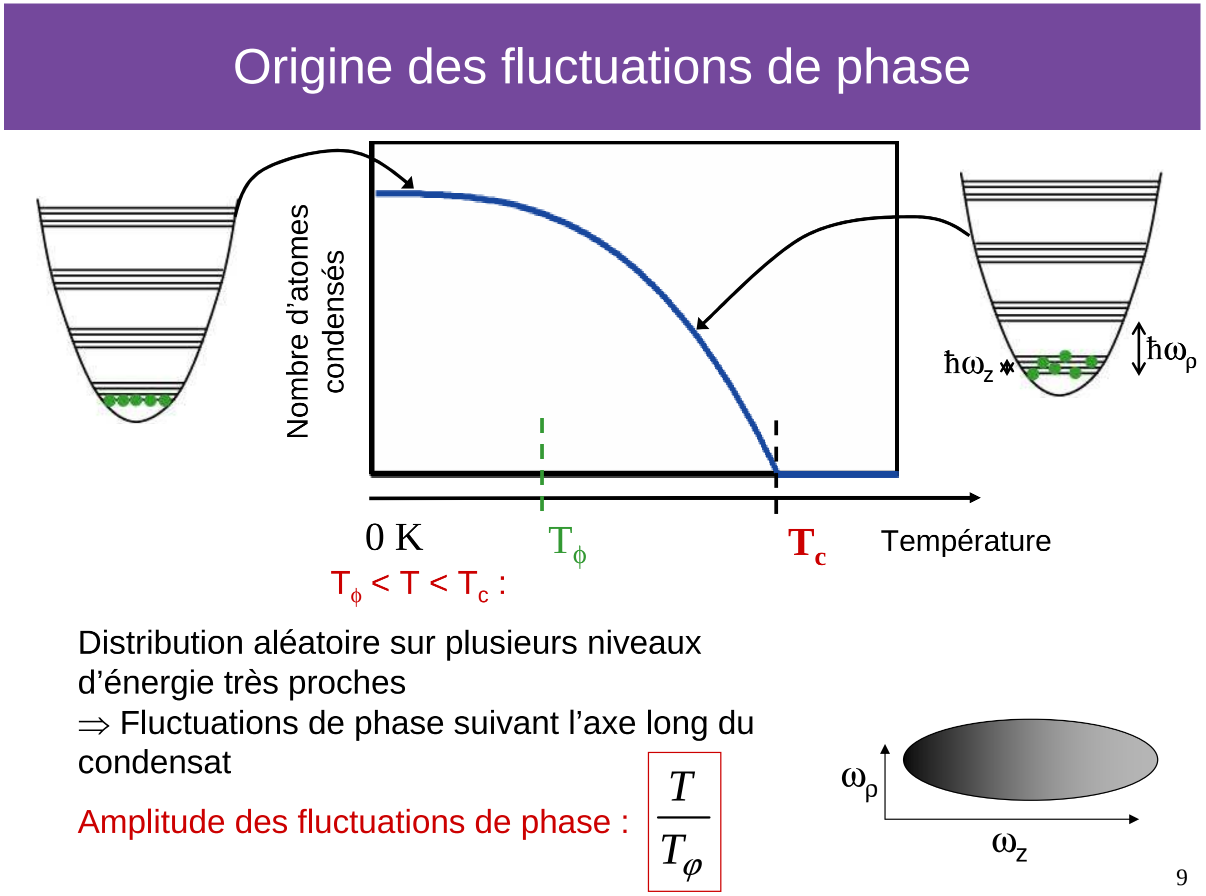
- **Main Theoretical Observation:** Finite Spectral Structure



2. Theoretical remarks about BEC and "Quasi-Condensate"

• The **first** tool is a Generalized Bose Condensation (GBC) *à la* van den Berg-Lewis-Pulé (1985-86):

$$\rho_{cond}(\beta) := \lim_{\delta \downarrow 0} \lim_{\Lambda} \sum_{j: E_j^{\omega} \leqslant \delta} \frac{1}{V} \langle N_{\Lambda}(\phi_j) \rangle_{H_{\Lambda}} > 0 \ , \ \beta > \beta_c = (k_B T_c)^{-1} \ .$$

N.B. In **highly anisotropic** boxes the perfect Bose-gas manifests a **type III** (or *non-extended*) GBC, when none of the one-particle states is macroscopically occupied:

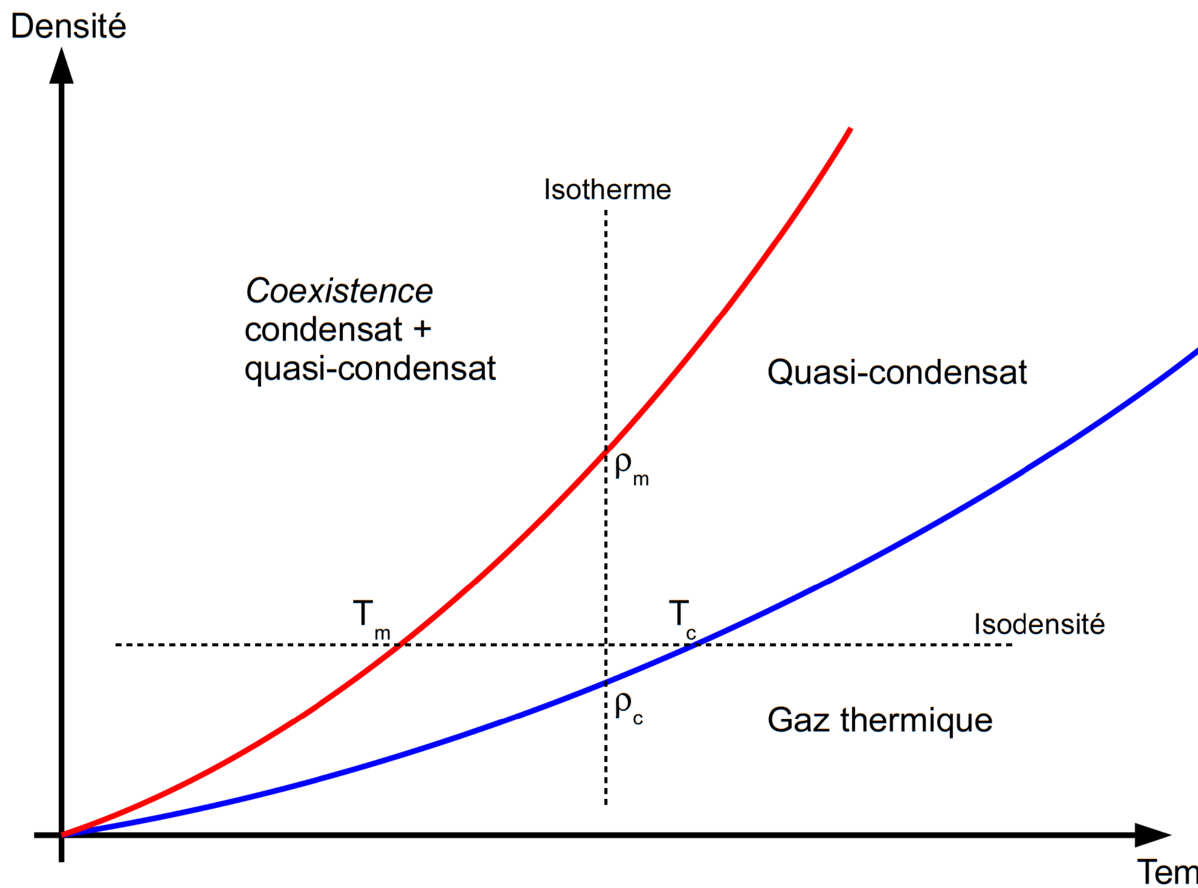
$$\lim_{\Lambda} \frac{1}{V} \langle N_{\Lambda}(\phi_j) \rangle_{H_{\Lambda}} = 0 \ , \ \text{but } \rho_0(\beta) > 0 \ .$$

• The **second** observation is that in 3D exponentially anisotropic in 2 directions boxes : $\Lambda_L = Le^{\alpha L} \times Le^{\alpha L} \times L$ there are **TWO** critical points for the perfect boson gas (PBG) condensation [van den Berg (1983)]:

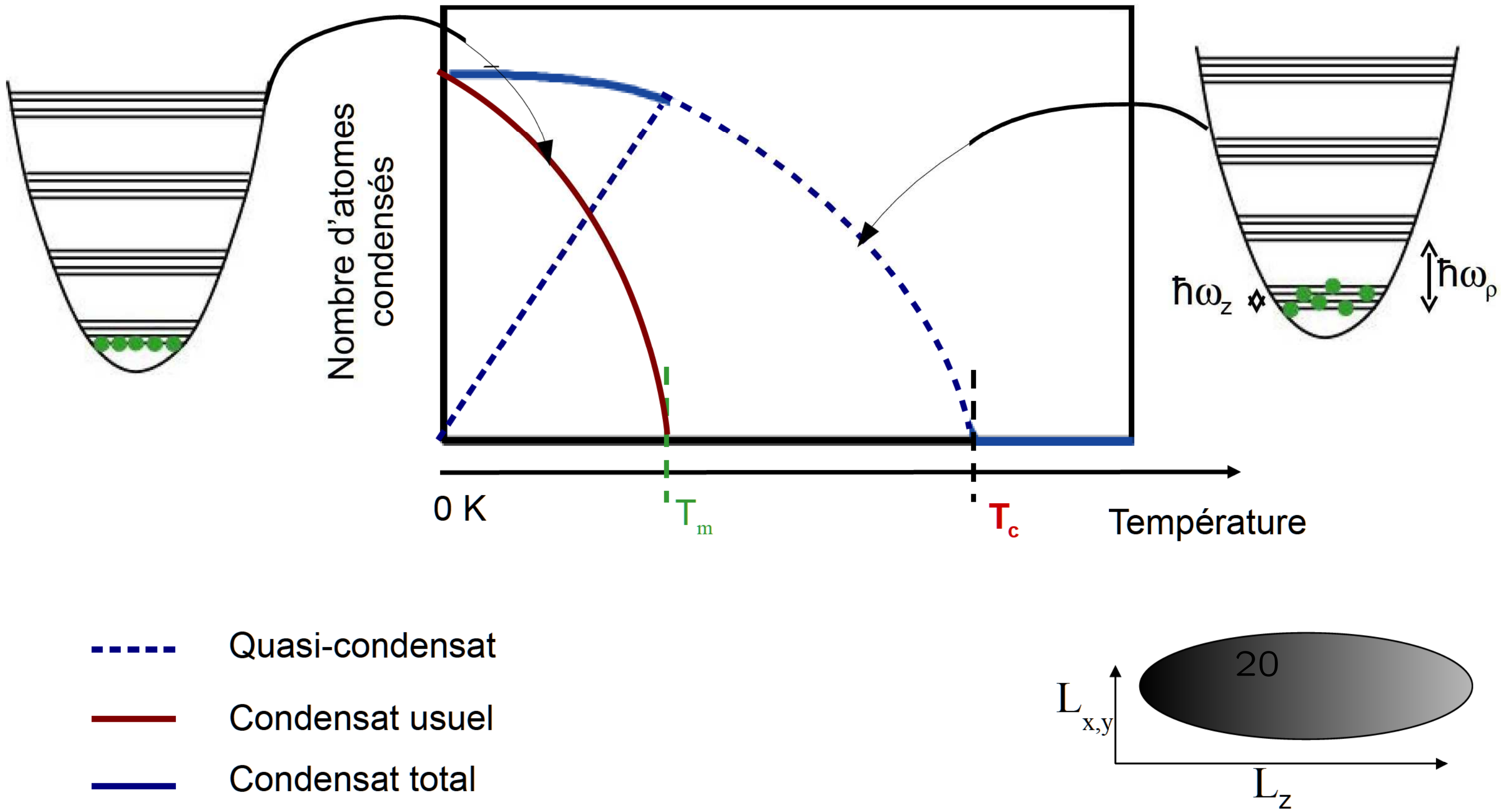
$$\rho_m(\beta) := \rho_c(\beta) + 2\alpha/\lambda_\beta^2 \ , \ \ \lambda_\beta = \hbar\sqrt{2\pi\beta/m} \ .$$

• For $\rho_c(\beta) < \rho < \rho_m(\beta)$ the GBC is *non-extensive* (type III): $\rho_{cond}^{III}(\beta, \rho) = \rho - \rho_c(\beta)$, with the *maximal* value: $\rho_m(\beta) - \rho_c(\beta)$.
• For $\rho_m(\beta) < \rho$ the GBC of the type I (conventional ground state BEC) **appears** with a density $\rho_{cond}^{I}(\beta) := \rho - \rho_m(\beta)$.
• So, for $\rho_m(\beta) < \rho$ the **saturated** *non-extensive* and the ground state condensates **coexist**:
$\rho_{cond}(\beta, \rho) = (\rho_m(\beta) - \rho_c(\beta)) + \rho_{cond}^{I}(\beta) = \rho - \rho_c(\beta)$.
• **Resumé:** "Quasi-Condensate" = Generalised BC of type III (**non-extensive** condensate) was known in Dublin since 1983-86!

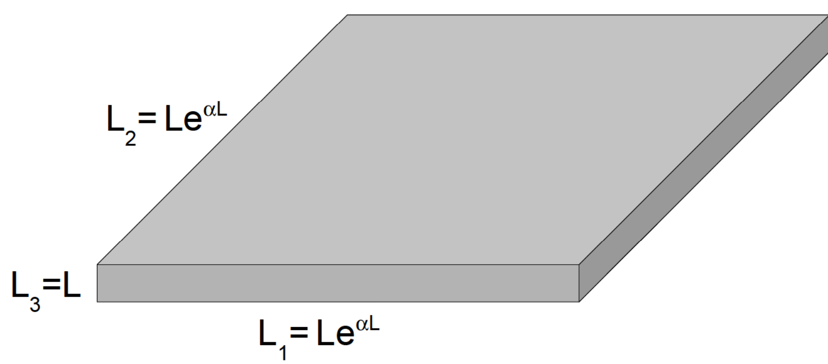
Diagramme de Phases



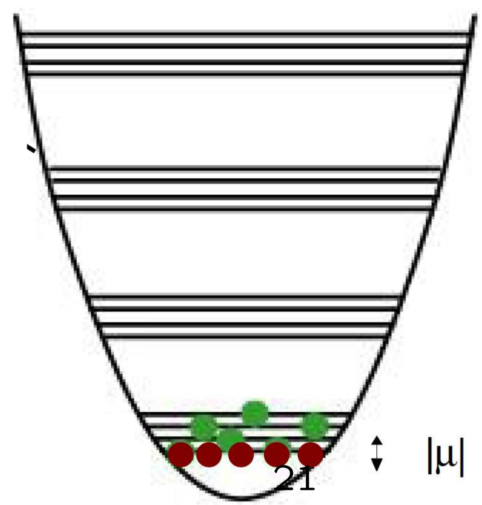
Fractions condensées:



Boite quasi-bi-dimensionnelle:



Modes condensées en k=0
et autour de k=0:



3. Conclusion: BEC in 3D exponentially anisotropic "cigar" traps is the "Quasi-Condensate" with two critical points

• **NB:** The van den Berg boxes $\Lambda = Le^{\alpha L} \times L \times L$, with **one-dimensional anisotropy** are **not** able to produce the *second* critical density $\rho_m(\beta) \neq \rho_c(\beta)$.

• **Theorem [Cond.Mat.Phys.(2010)]:** If 3D *harmonic trap* has a one-dimensional exponential anisotropy ("cigar"), then the *second* critical density $\rho_m(\beta) > \rho_c(\beta)$.

a. What is the traps thermodynamic limit? The *harmonic oscillator lengths* in three directions: $L_j := \sqrt{\hbar/(m\omega_j)}$, $j = 1, 2, 3$. Let $\omega_0 := (\omega_1\omega_2\omega_3)^{1/3}$ and N be a **total number** of the trapped bosons. Then the *proper thermodynamic limit* is $N \to \infty$, $\omega_0 \to 0$ (trap *opening*), with the **parameter** $n := N\omega_0^3$ *=constant*.

b. Then $n_c(\beta) < \infty$ and $T_c(n) = \hbar\, n^{1/3}/k_B\zeta(3)^{1/3}$.

c. Let $L_1 \geq L_2 = L_3 = L$ (almost isotropic trap). One gets a *ground-state* (type I) BC in the trap, when $n > n_c$:

$$\lim_{\omega_0 \to 0} N_{(0,0,0)}(\beta, \mu_{\omega_0})/N_{\omega_0}(\beta, \mu_{\omega_0}) = 1 - (T/T_c(n))^3 \ .$$

d. Let $L_1 = Le^{\gamma L^4}$, $\gamma > 0$ and $L_2 = L_3 = L$. Then the second **critical temperature** $T_m(n) < T_c(n)$:

$$T_m^3(n) + \tau^2 \ T_m(n) = T_c^3(n) \ , \ \tau = \hbar^2 \gamma^{1/2}/(mk_B \zeta(3)^{1/2}) > 0 \ .$$

e. Generalised BC of **type III** ("*quasi-condensate*"):

$$\lim_{N \to \infty} \frac{N_{gBEC}(\beta)}{N} = \begin{cases} 1 - (T/T_c)^3 \ , & T_m \leq T \leq T_c \ , \\ \tau^2 \ T/T_c^3 \ , & T \leq T_m \ . \end{cases}$$

f. Conventional *ground-state* condensation **(type I)** is:

$$\lim_{N\to\infty} \frac{N_{(0,0,0)}(\beta)}{N} = \begin{cases} 0\ , & T_m \leq T \leq T_c\ , \\ 1 - (T/T_c)^3(1 + \tau^2/T^2), & T \leq T_m\ , \end{cases}$$

g. Two condensates **coexist** for $T \leq T_m$:

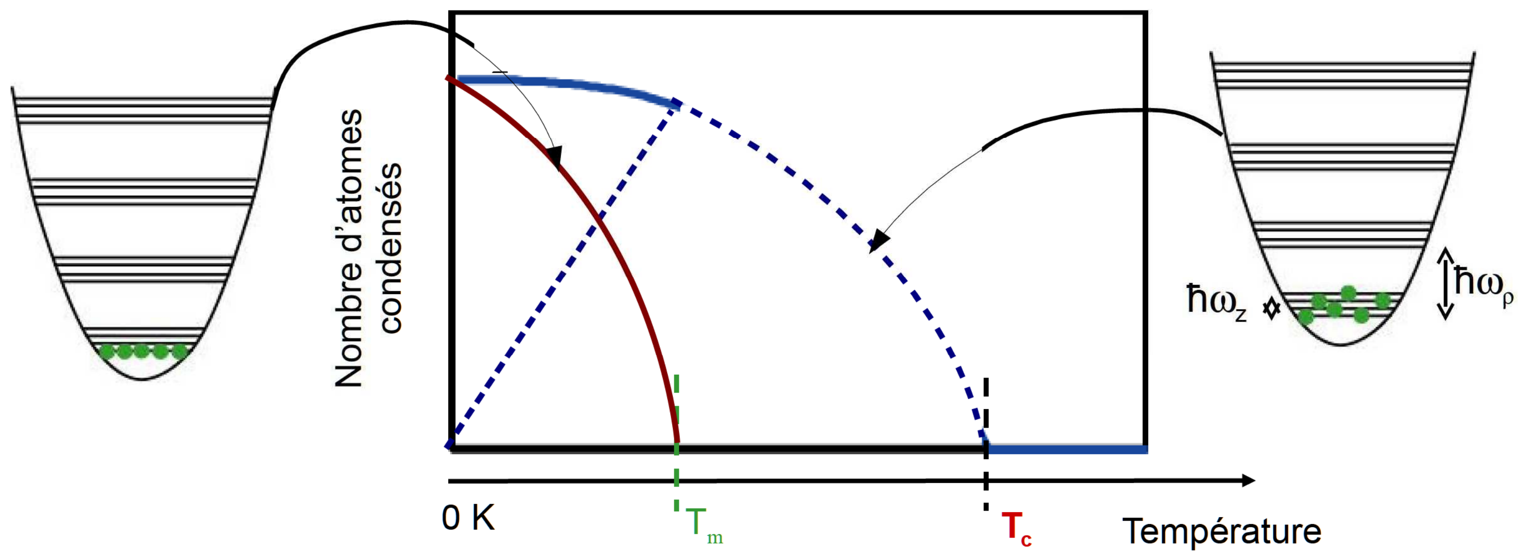
$$N_{totBEC}(\beta)/N := (N - N_c(\beta))/N =$$
$$(N_{gBEC}(\beta) + N_{(0,0,0)}(\beta))/N = 1 - (T/T_c)^3\ .$$

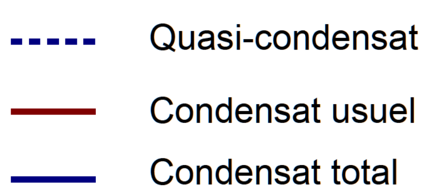
h. A "double" saturation mechanism:

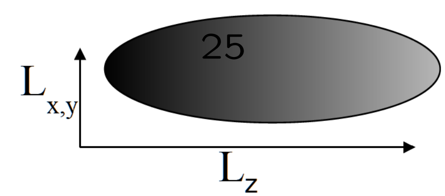
I. $N \mapsto N_{gBEC}$,

II. $N, N_{gBEC} \mapsto N_{(0,0,0)}$

Fractions condensées:







Thank you !